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ECE 204 *Numerical methods*

Successive parabolic interpolation for finding extrema

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Introduction

- In this topic, we will
  - Describe the method of successive parabolic interpolations for finding an extrema
  - Find a formula for the next point that is less subject to subtractive cancellation
  - Look at an example

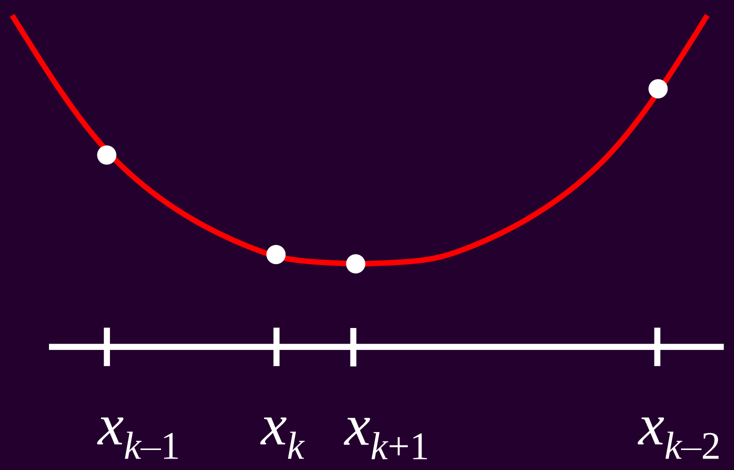
Interpolating quadratic polynomials

- Recall the secant method:
  - Given two points, let the next point be the root of the interpolating linear polynomial

- A straight line does not have an extrema,
  but a parabola does
  - Given three points, let the next point be the extremum of the interpolating quadratic polynomial

Interpolating quadratic polynomials

- For example:



Interpolating quadratic polynomials

- Given an initial approximations of the extremum x_0, x_1 and x_2, given the last three points, we find an interpolating quadratic $ax^2 + bx + c$ where

$$a = \frac{x_k\left(f(x_{k-1}) - f(x_{k-2})\right) + x_{k-1}\left(f(x_{k-2}) - f(x_k)\right) + x_{k-2}\left(f(x_k) - f(x_{k-1})\right)}{(x_k - x_{k-1})(x_{k-1} - x_{k-2})(x_{k-2} - x_k)}$$

  - If $a > 0$, it is a local minimum
  - If $a < 0$, it is a local maximum
  - If $a = 0$, we must examine the slope:
    - If the slope is zero, we may be very close to an extremum

$$b = \frac{x_k^2\left(f(x_{k-1}) - f(x_{k-2})\right) + x_{k-1}^2\left(f(x_{k-2}) - f(x_k)\right) + x_{k-2}^2\left(f(x_k) - f(x_{k-1})\right)}{(x_k - x_{k-1})(x_{k-1} - x_{k-2})(x_{k-2} - x_k)}$$

Interpolating quadratic polynomials

- The minimum is at $x = -b/2a$, or in other words:

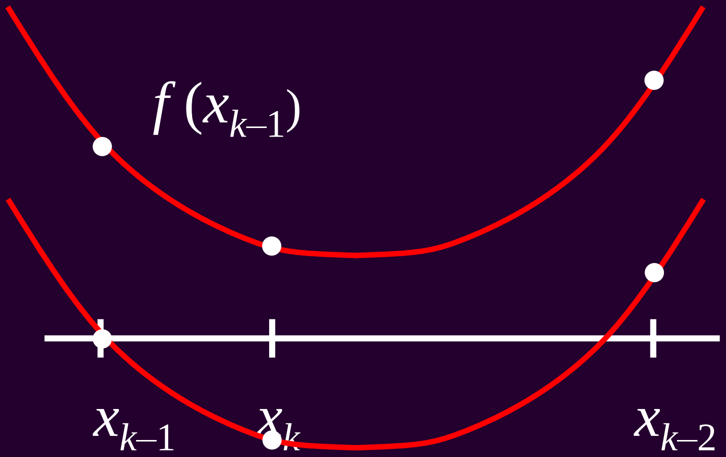
$$x_{k+1} \leftarrow \frac{1}{2} \frac{f(x_k)\left(x_{k-1}^2 - x_{k-2}^2\right) + f(x_{k-1})\left(x_{k-2}^2 - x_k^2\right) + f(x_{k-2})\left(x_k^2 - x_{k-1}^2\right)}{f(x_k)\left(x_{k-1} - x_{k-2}\right) + f(x_{k-1})\left(x_{k-2} - x_k\right) + f(x_{k-2})\left(x_k - x_{k-1}\right)}$$

  – Question: Is there anything we can do here to reduce subtractive cancellation?
    - Always arrange the previous three approximations so that

$$f(x_k) \le f(x_{k-1}) \le f(x_{k-2})$$

Interpolating quadratic polynomials

- The minimum of $f(x)$ is the same minimum of $f(x) - c$



Interpolating quadratic polynomials

- The location of the minimum is unchanged if we subtract off $f(x_{k-1})$ from the f values

$$x_{k+1} \leftarrow \frac{1}{2}\frac{\left(f(x_k)-f(x_{k-1})\right)\left(x_{k-1}^2-x_{k-2}^2\right)+\left(f(x_{k-2})-f(x_{k-1})\right)\left(x_k^2-x_{k-1}^2\right)}{\left(f(x_k)-f(x_{k-1})\right)\left(x_{k-1}-x_{k-2}\right)+\left(f(x_{k-2})-f(x_{k-1})\right)\left(x_k-x_{k-1}\right)}$$

  – Bonus: We calculated these differences when we found a

- Finally, we can factor out the average of the last two points:

$$x_{k+1} \leftarrow \frac{x_k+x_{k-1}}{2}+\frac{1}{2}\frac{\left(f(x_k)-f(x_{k-1})\right)\left(x_{k-1}-x_{k-2}\right)\left(x_{k-2}-x_k\right)}{\left(f(x_k)-f(x_{k-1})\right)\left(x_{k-1}-x_{k-2}\right)-\left(x_k-x_{k-1}\right)\left(f(x_{k-1})-f(x_{k-2})\right)}$$

Interpolating quadratic polynomials

- Question: Having found x_{k+1}, which point do we discard?
  - First, if $f(x_{k+1}) \geq f(x_{k-2})$, we have an issue, as either:
    - We were not sufficiently close to a minimum
    - The function is not sufficiently well behaved
  - Next, recall that $f(x_k) \leq f(x_{k-1}) \leq f(x_{k-2})$
  - If $\left| f(x_{k+1}) - f(x_k) \right| < \varepsilon_{\text{abs}}$ and $\left| x_{k+1} - x_k \right| < \varepsilon_{\text{step}}$, we are also finished: returner whichever is smaller
  - Otherwise, discard x_{k-2}

Rate of convergence

- The rate of convergence is super-linear, and appears to be

$$O\left(h^{1.3247}\right)$$

  - This value is close to the real root of $x^3 - x - 1$
    - Recall that the secant method was order $\frac{1}{2} + \frac{\sqrt{5}}{2}$,

      and this was a root of $x^2 - x - 1$

$$\frac{\sqrt[3]{100+12\sqrt{69}}}{6} + \frac{2}{\sqrt[3]{100+12\sqrt{69}}} \approx 1.324717957$$

Example

$x = 1.38629436112$

- Find the first minimum of $2e^{-2x} - e^{-x}$ starting with [1, 2]

1	1.5	2	1.507300561
–0.0972088747	–0.1235560234	–0.09870400542	
1.5	1.507300561	2	1.261931438
–0.1235560234	–0.1233763112	–0.09870400542	
1.261931438	1.5	1.507300561	1.394854544
–0.1228078935	–0.1235560234	–0.1233763112	
1.394854544	1.5	1.507300561	1.377481966
–0.1249909184	–0.1235560234	–0.1233763112	
1.377481966	1.394854544	1.5	1.386351606
–0.1249902067	–0.1249909184	–0.1235560234	–0.1249999996

Summary

- Following this topic, you now
  - Are aware the method of successive parabolic interpolations
    - This is similar to the secant method
  - Understand that it is possible to calculate the next point with less likelihood of subtractive cancellation
  - Know that it has super-linear convergence
  - Have seen an example

References

[1] https://en.wikipedia.org/wiki/Successive_parabolic_interpolation

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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